Equation (10) may now be compared with Prandtl's result for the infinite lifting line.[4] On the wing centerline $\bar{y} = 0$, reduction of Eq. (10) gives

$$(2\pi\hat{w}b_0)^{-1}\Gamma(0) = A(A+2)^{-1}J_0(\pi A b_0 k/4) + A\sum_{m=0}^{\infty}(A/2+2m)[(A/2+2m)^2-1]^{-1}\times J_{2m}(\pi A b_0 k/4) \quad (11)$$

where b_0 is the semichord measured at midspan. The analogous result for the infinitely long lifting line is[4]

$$(2\pi\hat{w}b_0)^{-1}|\Gamma(0)|_{A=\infty} = (1+\pi b_0 k/2)^{-1} \quad (12)$$

Figure 1 shows how the centerline circulation calculated from Eq. (11) approaches the limit Eq. (12) for large aspect ratio.

References

[1] Ashley, H. and Landahl, M., *Aerodynamics of Wings and Bodies*, Addison-Wesley, Reading, Mass., 1965, Chap. 7,pp. 137–142.

[2] Abramowitz, M. and Stegun, I. A., eds., *Handbook of Mathematical Functions*, Dover, New York, 1965.

[3] Heaslet, M. A. and Spreiter, J. R., "Reciprocity Relations in Aerodynamics," TN 2700, 1952, NACA.

[4] Prandtl, L. and Betz, A., "Vier Abhandlungen zur Hydrodynamik und Aerodynamik," *Im Selbstverlag des Kaiser Wilhelm-Instituts für Strömungsforschung,* Göttingen, 1927.

Turbulent Skin Friction for Tapered Wings

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Nomenclature

b	= wing span
$c(\xi)$	= wing chord at spanwise station
c_f	= average skin-friction coefficient
c_{f_0}	= average skin-friction coefficient for zero taper ratio
$c_f(\xi)$	= skin-friction coefficient at spanwise station
c_r	= wing root chord
c_t	= wing tip chord
R	= Reynolds number $[U_\infty c(\xi)/\nu]$
R_r	= Reynolds number based on root chord
U_∞	= freestream velocity
x	= spanwise station
ξ	= $x/(b/2)$
λ	= taper ratio (c_t/c_r)
ν	= kinematic viscosity
$\log_{10}(\ )$	= logarithm to base 10
$\ln(\ )$	= logarithm to base e

THE problems involved in a rapid determination of the skin-friction drag for tapered wings have been discussed by both E. J. Hopkins[1] and A. Barkhem.[2] Very simply, the main problem resolves to one of estimating the over-all skin friction using a Reynolds number based on some average chord, usually the wing mean aerodynamic chord, which purports to represent the entire wing. This method is rapid but may result in small but significant errors (e.g., 3–4% for highly tapered wings at low Reynolds numbers). The alternative is to perform a spanwise integration of the local skin-friction coefficient. This is accurate but becomes simple only if the skin-friction formula used is simple.

The method developed below is both simple to use and very accurate in its results. It is derived on the basis of Reynolds numbers referred to the wing root-chord that are of practical interest in aircraft design.

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From the Prandtl-Schlichting turbulent skin-friction formula for a smooth flat plate, the local skin friction is[3]

$$c_f(\xi) = 0.455/(\log_{10}R)^{2.58} \quad (1)$$

The general planform considered is shown in Fig. 1. The local chord can be expressed as a function of planform geometry as

$$c(\xi)/c_r = 1 - (1-\lambda)\xi \quad (2)$$

Considering the Reynolds number, we find

$$R = U_\infty c/\nu = (U_\infty c_r/\nu)[1-(1-\lambda)\xi] = R_r[1-(1-\lambda)\xi] \quad (3)$$

Substitution of Eq. (3) into Eq. (1) gives the local skin-friction coefficient as a function of the root-chord Reynolds number and planform geometry. Thus,

$$c_f(\xi) = 0.455/\{\log_{10}R_r[1-(1-\lambda)\xi]\}^{2.58} \quad (4)$$

We can define the average skin-friction coefficient/unit area of surface by

$$c_f = \left[\int_0^1 c_f(\xi)c(\xi)d\xi\right]\Big/\left[\int_0^1 c(\xi)d\xi\right] \quad (5)$$

One can then write

$$c_f = \left[\int_0^1 c_f(\xi)c(\xi)/c_r d\xi\right]\Big/\left[\int_0^1 c(\xi)/c_r d\xi\right] \quad (6)$$

which with Eqs. (2) and (4) gives

$$c_f = \frac{2}{1+\lambda}\frac{0.455}{[\log_{10}R_r]^{2.58}}\times\int_0^1\frac{1-(1-\lambda)\xi d\xi}{\{1+\ln[1-(1-\lambda)\xi]/\ln R_r\}^{2.58}} \quad (7)$$

Expanding the denominator in an infinite series gives

$$c_f = \frac{2}{1+\lambda}\frac{0.455}{(\log_{10}R_r)^{2.58}}\int_0^1[1-(1-\lambda)\xi]\times\{1-2.58\ln[1-(1-\lambda)\xi]/\ln R_r + 4.6182[\ln[1-(1-\lambda)\xi]/\ln R_r]^2+\dots\}d\xi \quad (8)$$

It is easily shown that this integrand series is convergent as long as

$$\lambda > 1/R_r \quad (9)$$

For all practical purposes, this puts no restriction on λ for moderate to large R_r. Thus, integrating Eq. (8) term by

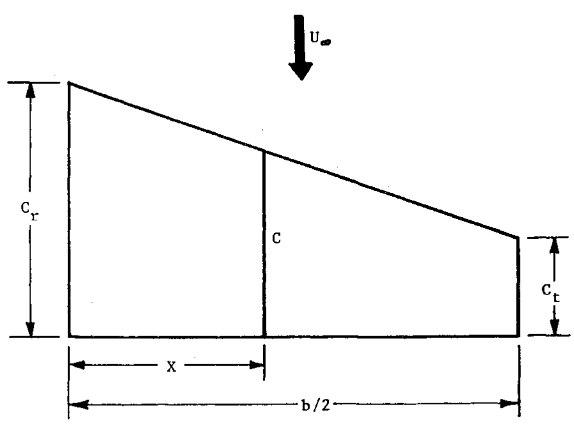

Fig. 1 Wing planform geometry.

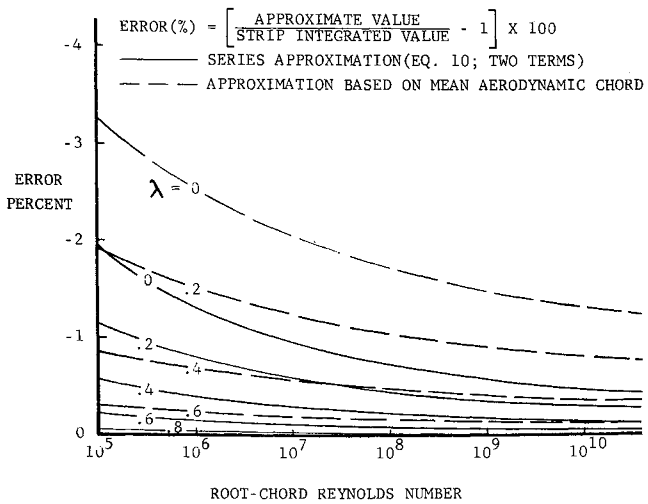

Fig. 2 Error in calculated skin friction as compared with strip integrated values.

term, we obtain

$$c_f = \frac{0.455}{(\log_{10}R_r)}\left\{1 + \frac{2.58}{(1+\lambda)\ln R_r} \times \left(\frac{1+\lambda}{2} + \frac{\lambda^2}{1-\lambda}\ln\lambda\right) + \frac{2.3091}{(\ln R_r)^2} \times \left[2\frac{\lambda^2}{1-\lambda^2}(1-\ln\lambda)\ln\lambda + 1\right] + \ldots\right\} \quad (10)$$

If one places a further restriction on λ, it would be possible to make the series representation for c_f converge very quickly, such that only a few terms of Eq. (10) need be retained for good accuracy. If, in fact

$$\lambda \gg 1/R_r \quad (11)$$

the desired results are obtained. Again this places little restriction on λ.

From Eq. (10) it is seen that for rectangular planforms (i.e., $\lambda = 1$), c_f has the proper limit value. If $\lambda \to 0$, the relation obtained from Eq. (10) is exactly the same as obtained from the integral equation (7) for $\lambda = 0$ and is given by

$$c_{f_0} = \frac{0.455}{(\log_{10}R_r)^{2.58}}[1 + 1.29/\ln R_r + 2.3091/(\ln R_r)^2 + \ldots] \quad (12)$$

Strip integrated values for c_f were computed by using Eq. (7) with 100 strips/semispan. These values were used as a basis for comparison with both the approximate method developed herein [Eq. (10) or (12)] and the standard method using the wing mean aerodynamic chord. For Reynolds numbers in the range 10^5–10^{10}, the first two terms of Eq. (10) [or Eq. (12) for $\lambda = 0$] differ from strip integrated values by less than 2% for all taper ratios and in addition give better accuracy than the method utilizing the wing mean aerodynamic chord for all taper ratios except for $\lambda = 1$, where they are both exact (Fig. 2).

If three terms of Eq. (10) or Eq. (12) are used, the series approximation is nearly exact; differing from strip integrated values by only 0.4% for the worst case.

References

[1] Hopkins, E. J., "Some Effects of Planform Modification on the Skin Friction Drag," *AIAA Journal*, Vol. 2, No. 2, Feb. 1964, pp. 413–414.

[2] Barkhem, A., "Skin-Friction Formula for Tapered and Delta Wings," *Journal of Aircraft*, Vol. 6, No. 3, May-June 1969, p. 284.

[3] Schlichting, H., *Boundary Layer Theory*, 6th ed., McGraw-Hill, New York, 1968, p. 602.

Convergence-Proof of Discrete-Panel Wing Loading Theories

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Introduction

MANY wing surface loading theories use a finite elemental grid or lattice representation for the loading integral. As the lattice panel, such as a elemental horseshoe vortex, becomes small it induces velocity on its neighbor that approaches infinity. It had not been mathematically shown that this limit product of zero and infinity converges to the accurate finite loading.

The chordwise section-loading integral equation can be written in terms of vortices equally spaced along the wing chord. The summed vortex induced downwash at midpoints between the vortices give equations that satisfy the boundary condition of no flow through the wing. These result in N unknowns (of N load-vortices) and N equations which can be solved simultaneously.

A solution for any value of N has been derived which since 1967 has been proven accurate and useful for defining the chordwise loading curve from constant loading segments. Having exact mathematical solutions for a large number of simultaneous equations permits making accuracy checks of computers. Actually, the present solution was derived because a computer 15-term simultaneous equations solution appeared a little doubtful. This computer was proven to give erroneous loading values of up to 14% near the leading edge.

Analysis

The downwash at x_{em} due to a vortex at x_n is

$$w_{mn} = \Gamma_n/2\pi(x_{em} - x_n) \quad (1)$$

The distribution of chordwise vortices at the $\frac{1}{4}$ chord point of the equal length panels are shown in Fig. 1.

Let N be the number of panels. Then c/N is the length of each equal-length panel. The circulation at the quarter-chord of the nth panel is at the chord station

$$x_n = (n-1)(c/N) + (c/4N) = [n - \tfrac{3}{4}](c/N) \quad (2)$$

The $\frac{3}{4}$ chord of the nth panel is at the chord station

$$x_{em} = (m-1)(c/N) + (3c/4N) = (m - \tfrac{1}{4})(c/N) \quad (3)$$

With Eqs. (2) and (3), Eq. (1) becomes

$$(w_{mn}/V) = (N\Gamma_n/\pi c V)/(2m - 2n + 1) \quad (4)$$

Define e_n as

$$e_n = N\Gamma_n/\pi c V\alpha = \gamma_n/\pi V\alpha = -\Delta C_p/2\pi\alpha \quad (5)$$

Then the total downwash at chord station m is the summation of all the vortices. The flow angle that satisfies the boundary condition of no flow through the wing is from Eq. (4) given by

$$(\alpha_m/\alpha) = \sum_{n=1}^{N} e_n/(2m - 2n + 1);\ m = 1, 2, \ldots, N \quad (6)$$

where α_m is the angle or slope of the wing camber line at chord station m.

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